TECHNICAL NOTE R-66

ANALYSIS OF THE RELATIVE LIQUID LEVELS IN TWO INTERCONNECTED TANKS WITH DIFFERENT FLOW RATES

Part I - Development of Equations for Application to Space Flight and Non-Flight Conditions

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ABSTRACT

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The purpose of the investigation was to provide a means for calculating the mass and liquid level difference between two tanks interconnected by a tube. The influence of a hydraulic loss in the interconnecting line is taken into account. The nonlinear differential equations are applicable not only for flight draining but also for the filling and emergency draining processes, even when the port is located on a single tank.

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LIST OF SYMBOLS

A	Cross-sectional area
b	Minor semi-axis of the oblate hemispheroid part of the tank
F	Axial thrust acting on the vehicle
g_e	Gravitational constant on the earth
g_t	Acceleration of the vehicle at any time t
h	Height of the liquid level in the tank measured from the bottom
K	Length of the cylindrical part of the tank plus the minor semi-axis of the oblate hemispheroid
m	Instantaneous mass of fluid in the tank
$\dot{m}$	Mass flow rate per second
p	Static pressure
q	Hydraulic loss factor, defined by $q = \frac{\Delta p}{\frac{1}{2} \rho v_m^2}$ (See Appendix)
R	Radius of the tank
r	Radius of the interconnect line
r_e	Radius of the fill and emergency drain port
t	Time
V	Volume
v	Velocity
v_m	Mean velocity of fluid in the interconnect line
W	Weight of the vehicle
α	Ratio of total propellant flow rate to the initial weight of the vehicle

γ	Weight density
ρ	Mass density
$\dot{\omega}$	Weight flow rate per second

Subscripts

0	Refers to condition t = 0
1	Refers to tank 1
10	Refers to tank 1 at t = 0
1(t)	Refers to tank 1 at each time t
1U	Refers to used propellant flow rate out of tank 1
2	Refers to tank 2
20	Refers to tank 2 at t = 0
2(t)	Refers to tank 2 at each time t
2U	Refers to used propellant flow rate out of tank 2
e	Refers to exit conditions
I	Refers to interconnect line
L	Refers to remaining weight of propellant at cut-off
P	Refers to total propellant

INTRODUCTION

When multiple propellant tanks are used in a propulsion system, there may be a residual fluid remaining in one tank at cut-off due to differences in flow rates. This would result from performance and mixture ratio deviations in the different engines. To reduce this problem, a line connecting the tanks can be used. However, due to the hydraulic friction in this line, a certain liquid level height difference or a pressure difference must be applied to overcome this loss.

In the analysis, the tube size and the hydraulic factor are interdependent. Both have a large influence on the level differences of the tanks. Equations have been developed for a generalized analysis of two tanks and an interconnecting line. The equations are applicable to any system using this configuration for an outer space flight and a non-flight terrestrial condition. The geometry of the tank shape consisting of a cylindrical part and an upper and lower oblate hemispheroidal section is included. The equations describing the process are coded in a FORTRAN program using the Runge-Kutta method for solving the nonlinear differential equations. This program is described in Reference 1.

Part II of this report contains the results for the Multi-Mission Module LI, LII, and SVI stages.[2]

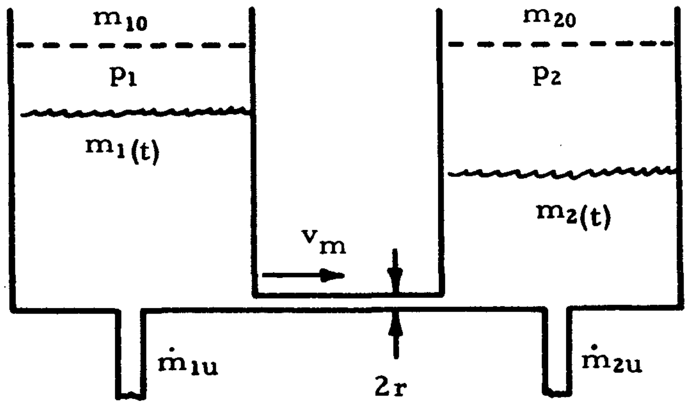
Acknowledgement is made of the assistance of Dr. F. W. Geiger in the theoretical analysis and of B. H. Kavanaugh, Jr. in the programming of the equations for the computer.

DEVELOPMENT OF EQUATIONS

IN-FLIGHT OPERATION

General

The system to be evaluated consists of two tanks joined by an interconnect line as illustrated in the sketch below.



m_{10} - initial mass of fluid in tank 1

m_{20} - initial mass of fluid in tank 2

$\dot{m}_{1u}$ - used mass flow rate out of tank 1

$\dot{m}_{2u}$ - used mass flow rate out of tank 2

The mass of fluid at any time in the tanks is

$$m_1(t) = m_{10} - \int_0^t \dot{m}_1 \, dt \quad ,$$
$$m_2(t) = m_{20} - \int_0^t \dot{m}_2 \, dt \quad , \qquad (1)$$

where the integrals represent the total mass flown out of the tank consisting of the used mass and the mass through the interconnecting line. With

$$\dot{m}_1 = \dot{m}_{1u} + \rho\, v_m\, \pi r^2 \,,$$
$$\dot{m}_2 = \dot{m}_{2u} - \rho\, v_m\, \pi r^2 \,, \tag{2}$$

the mass is

$$m_1(t) = m_{10} - \int_0^t (\dot{m}_{1u} + \rho\, v_m\, \pi r^2)\, dt \,,$$
$$m_2(t) = m_{20} - \int_0^t (\dot{m}_{2u} - \rho\, v_m\, \pi r^2)\, dt \,, \tag{3}$$

or the difference

$$m_1(t) - m_2(t) = m_{10} - m_{20} - \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt - \int_0^t 2\, \rho\, v_m\, \pi r^2\, dt \,. \tag{4}$$

The mean velocity v_m in the interconnecting line is a function of the level difference, pressure difference and the hydraulic losses in the line.

$$\frac{\rho}{2}\, v_m{}^2 \cdot q = \Delta p = (p_1 - p_2) + \rho\, g_{(t)}\, (h_{1(t)} - h_{2(t)}) \,. \tag{5}$$

The acceleration on the vehicle increases with time due to the changing of the total mass at constant thrust.

$$g_{(t)} = \frac{F}{m_o\left(1 - \frac{\dot{m}_p}{m_o}\, t\right)} = \frac{F}{m_o\,(1 - a\,t)}\ , \tag{6}$$

where

m_o = initial mass of the vehicle,

$\dot{m}_p$ = constant propellant flow rate.

So, the mean velocity is

$$v_m = \left[\frac{2\,\dfrac{F}{m_o\,(1 - a\,t)}\,(h_{1(t)} - h_{2(t)}) + 2\,(p_1 - p_2)/\rho}{q}\right]^{\frac{1}{2}} . \tag{7}$$

Substitution into Equation (4) gives

$$m_{1(t)} - m_{2(t)} = m_{10} - m_{20} - \int_o^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt$$

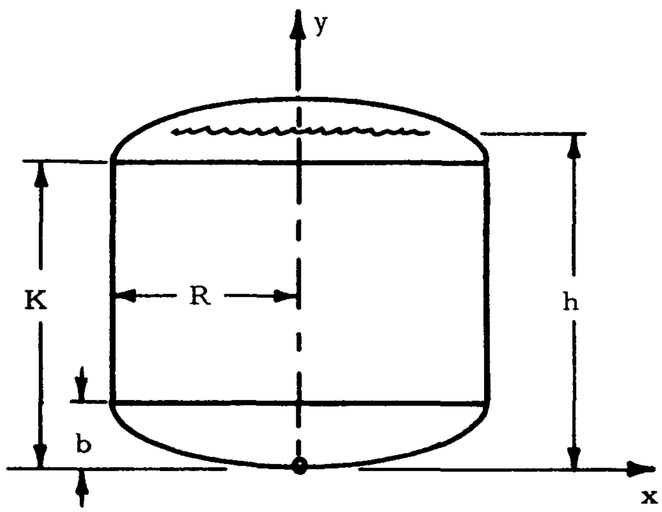
$$- 2\,\pi\,\rho\, r^2 \int_o^t \left[\frac{2\,\dfrac{F}{m_o\,(1 - a\,t)}\,(h_{1(t)} - h_{2(t)}) + 2\,(p_1 - p_2)/\rho}{q}\right]^{\frac{1}{2}} dt\ . \tag{8}$$

Because there are two unknowns a second equation is needed. The necessary equation comes from a mass balance on the two tanks taken as a single system.

$$m_{1(t)} + m_{2(t)} = m_{10} + m_{20} - \int_o^t (\dot{m}_{1u} + \dot{m}_{2u})\, dt\ . \tag{9}$$

Upper Oblate Hemispheroid

Additionally, a geometrical relation between the fluid height and the volume must be found. The fluid level can be in the upper oblate hemispheroid, the cylindrical part or the lower oblate hemispheroid.



For h > K the volume in the upper oblate hemispheroid is

$$V = \pi \int_K^h x^2 \, dy \; .$$

The equation of the ellipse is

$$\frac{x^2}{R^2} + \frac{(y - K)^2}{b^2} = 1 \; ,$$

$$V = \pi \int_K^h R^2 \left(1 - \frac{(y - K)^2}{b^2}\right) dy \; ,$$

$$V = \pi R^2 \left[h - K - \frac{h^3 - K^3}{3b^2} + \frac{Kh^2 - K^3}{b^2} - \frac{K^2 h - K^3}{b^2} \right] \; ,$$

or simplified

$$V = \pi R^2 \left[h - K - \frac{(h - K)^3}{3b^2} \right] .$$

If $h > K$ the volume in the tank is

$$V = \pi R^2 \left[h - K - \frac{(h - K)^3}{3b^2} \right] + \pi R^2 (K - b) + \frac{2}{3} \pi R^2 b .$$

For h_1, $h_2 > K$ the mass difference is

$$m_{1(t)} - m_{2(t)} = \rho \pi R^2 \left[h_{1(t)} - h_{2(t)} - \frac{(h_{1(t)} - K)^3}{3b^2} + \frac{(h_{2(t)} - K)^3}{3b^2} \right] . \tag{10}$$

With Equation (8) the relation is

$$\rho \pi R^2 \left[h_{1(t)} - h_{2(t)} - \frac{(h_{1(t)} - K)^3}{3b^2} + \frac{(h_{2(t)} - K)^3}{3b^2} \right] = m_{10} - m_{20} - \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt$$

$$- 2 \pi \rho r^2 \int_0^t \left[\frac{2 \frac{F}{m_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} dt , \tag{11}$$

or

$$h_{1(t)} - h_{2(t)} - \frac{(h_{1(t)} - K)^3}{3b^2} + \frac{(h_{2(t)} - K)^3}{3b^2} = \frac{1}{\rho \pi R^2} (m_{10} - m_{20}) - \frac{1}{\rho \pi R^2} \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt$$

$$- 2 \frac{r^2}{R^2} \int_0^t \left[\frac{2 \frac{F}{m_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} dt .$$

Differentiation of Equation (11) with respect to time gives

$$\frac{dh_{1(t)}}{dt}\left(1-\frac{(h_{1(t)}-K)^2}{b^2}\right)-\frac{dh_{2(t)}}{dt}\left(1-\frac{(h_{2(t)}-K)^2}{b^2}\right)=-\frac{1}{\rho\,\pi\,R^2}\,(\dot{m}_{1u}-\dot{m}_{2u})$$

$$-2\,\frac{r^2}{R^2}\left[\frac{2\,\frac{F}{m_o\,(1-a\,t)}\,(h_{1(t)}-h_{2(t)})+2\,(p_1-p_2)/\rho}{q}\right]^{\frac{1}{2}}. \tag{12}$$

The derivative $\frac{dh_1}{dt}$ is

$$\frac{dh_1}{dt}=\frac{1}{\left(1-\frac{(h_{1(t)}-K)^2}{b^2}\right)}\left\{\frac{dh_2}{dt}\left(1-\frac{(h_{2(t)}-K)^2}{b^2}\right)-\frac{1}{\rho\,\pi\,R^2}\,(\dot{m}_{1u}-\dot{m}_{2u})\right.$$

$$\left.-2\,\frac{r^2}{R^2}\left[\frac{2\,\frac{F}{m_o\,(1-a\,t)}\,(h_{1(t)}-h_{2(t)})+2\,(p_1-p_2)/\rho}{q}\right]^{\frac{1}{2}}\right\}. \tag{13}$$

Substitution of the relations for $m_{1(t)}$ and $m_{2(t)}$, when the fluid level is in the upper oblate hemispheroid, into Equation (9) yields

$$\rho\left\{\pi\,R^2\left[h_{1(t)}+h_{2(t)}-\frac{(h_{1(t)}-K)^3}{3b^2}-\frac{(h_{2(t)}-K)^3}{3b^2}-2K\right]+2\,\pi\,R^2\,(K-b)+\frac{4}{3}\pi\,R^2\,b\right\}$$

$$=m_{10}+m_{20}-\int_o^t(\dot{m}_{1u}+\dot{m}_{2u})\,dt\ , \tag{14}$$

or

$$h_1(t) + h_2(t) - \frac{(h_1(t) - K)^3}{3b^2} - \frac{(h_2(t) - K)^3}{3b^2} - \frac{2}{3} b = \frac{1}{\rho \pi R^2} (m_{10} + m_{20})$$

$$- \frac{1}{\rho \pi R^2} \int_0^t (\dot{m}_{1u} + \dot{m}_{2u})\, dt \, .$$

Differentiation of Equation (14) with respect to time gives

$$\frac{dh_1}{dt}\left(1 - \frac{(h_1(t) - K)^2}{b^2}\right) + \frac{dh_2}{dt}\left(1 - \frac{(h_2(t) - K)^2}{b^2}\right) = - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} + \dot{m}_{2u}) \, . \tag{15}$$

After solving for $\frac{dh_2}{dt}$ the result is

$$\frac{dh_2}{dt} = \frac{1}{\left(1 - \frac{(h_2(t) - K)^2}{b^2}\right)} \left[- \frac{dh_1}{dt}\left(1 - \frac{(h_1(t) - K)^2}{b^2}\right) - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} + \dot{m}_{2u}) \right] . \tag{16}$$

Equations (13) and (16) are the two differential equations for the upper oblate spheroid part of the tank.

Cylindrical Section

From Equations (8) and (9) the relations for the cylindrical part are

$$h_1(t) - h_2(t) = \frac{1}{\rho \pi R^2} (m_{10} - m_{20}) - \frac{1}{\rho \pi R^2} \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt$$

(17)

$$- 2 \frac{r^2}{R^2} \int_0^t \left[\frac{2 \frac{F}{m_o (1 - \alpha t)} (h_1(t) - h_2(t)) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} dt \, ,$$

and

$$h_{1(t)} + h_{2(t)} = -\frac{2}{3}\, b + \frac{1}{\rho \pi R^2}\,(m_{10} + m_{20}) - \frac{1}{\rho \pi R^2}\int_0^t (\dot{m}_{1u} + \dot{m}_{2u})\, dt\,. \tag{18}$$

Differentiating Equations (17) and (18) with respect to time

$$\frac{dh_1}{dt} - \frac{dh_2}{dt} = -\frac{1}{\rho \pi R^2}\,(\dot{m}_{1u} - \dot{m}_{2u}) - 2\,\frac{r^2}{R^2}\left[\frac{2\,\frac{F}{m_o\,(1 - \alpha t)}\,(h_{1(t)} - h_{2(t)}) + 2\,(p_1 - p_2)/\rho}{q}\right]^{\frac{1}{2}}, \tag{19}$$

or with the $\frac{dh_1}{dt}$ explicit

$$\frac{dh_1}{dt} = \frac{dh_2}{dt} - \frac{1}{\rho \pi R^2}\,(\dot{m}_{1u} - \dot{m}_{2u}) - 2\,\frac{r^2}{R^2}\left[\frac{2\,\frac{F}{m_o\,(1 - \alpha t)}\,(h_{1(t)} - h_{2(t)}) + 2\,(p_1 - p_2)/\rho}{q}\right]^{\frac{1}{2}},$$

and

$$\frac{dh_1}{dt} + \frac{dh_2}{dt} = -\frac{1}{\rho \pi R^2}\,(\dot{m}_{1u} + \dot{m}_{2u})\,, \tag{20}$$

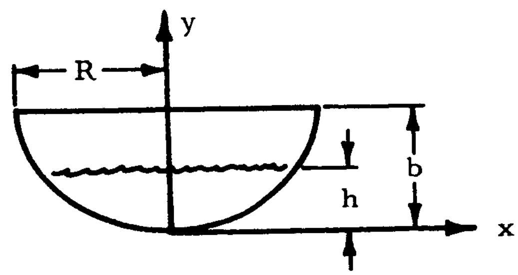
or with the $\frac{dh_2}{dt}$ explicit

$$\frac{dh_2}{dt} = -\frac{dh_1}{dt} - \frac{1}{\rho \pi R^2}\,(\dot{m}_{1u} + \dot{m}_{2u})\,.$$

Equations (19) and (20) are the two differential equations of $h_{1(t)}$ and $h_{2(t)}$ for the cylindrical part of the tank.

<u>Lower Oblate Hemispheroid</u>

Now two equations for the lower oblate hemispheroid part of the tank are necessary.



For h < b the volume in the oblate spheroid at the station h is

$$V = \frac{2}{3}\pi R^2 b - \pi \int_h^b x^2 \, dy ,$$

$$V = \frac{2}{3}\pi R^2 b - \pi R^2 \int_h^b \left(1 - \frac{(y-b)^2}{b^2}\right) dy ,$$

$$V = \frac{2}{3}\pi R^2 b - \pi R^2 \left[y \Big|_h^b - \frac{y^3}{3b^2} \Big|_h^b + \frac{1}{b} y^2 \Big|_h^b - y \Big|_h^b \right] ,$$

$$V = \pi R^2 \left[\frac{h^2}{b} - \frac{h^3}{3b^2} \right] .$$

If the fluid levels $h_{1(t)}$ and $h_{2(t)}$ are in the section $h \leq b$ the mass difference is

$$m_{1(t)} - m_{2(t)} = \rho \pi R^2 \left[\frac{h_{1(t)}^2 - h_{2(t)}^2}{b} - \frac{h_{1(t)}^3 - h_{2(t)}^3}{3b^2} \right] , \tag{21}$$

and the mass in both tanks is

$$m_{1(t)} + m_{2(t)} = \rho \pi R^2 \left[\frac{h_{1(t)}^2 + h_{2(t)}^2}{b} - \frac{h_{1(t)}^3 + h_{2(t)}^3}{3b^2} \right] . \tag{22}$$

Application of Equations (8) and (21) gives

$$\begin{aligned} \rho \pi R^2 \left[\frac{h_{1(t)}^2 - h_{2(t)}^2}{b} - \frac{h_{1(t)}^3 - h_{2(t)}^3}{3b^2} \right] &= m_{10} - m_{20} - \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt \\ &\quad - 2 \pi \rho r^2 \int_0^t \left[\frac{2 \dfrac{F}{m_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} dt , \end{aligned} \tag{23}$$

or

$$\begin{aligned} \frac{h_{1(t)}^2 - h_{2(t)}^2}{b} - \frac{h_{1(t)}^3 - h_{2(t)}^3}{3b^2} &= \frac{1}{\rho \pi R^2} (m_{10} - m_{20}) - \frac{1}{\rho \pi R^2} \int_0^t (\dot{m}_{1u} - \dot{m}_{2u})\, dt \\ &\quad -2 \frac{r^2}{R^2} \int_0^t \left[\frac{2 \dfrac{F}{m_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} dt , \end{aligned}$$

Differentiation of Equation (23) with respect to time yields

$$\frac{2}{b} h_1(t) \frac{dh_1}{dt} - \frac{2}{b} h_2(t) \frac{dh_2}{dt} - \frac{1}{b^2} h_1(t)^2 \frac{dh_1}{dt} + \frac{1}{b^2} h_2(t)^2 \frac{dh_2}{dt} \tag{24}$$

$$= - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} - \dot{m}_{2u}) - 2 \frac{r^2}{R^2} \left[\frac{2 \frac{F}{m_o (1 - at)} (h_1(t) - h_2(t)) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} ,$$

or with the $\frac{dh_1}{dt}$ explicit

$$\frac{dh_1}{dt} = \frac{1}{\left(\frac{2}{b} h_1(t) - \frac{1}{b^2} h_1(t)^2\right)} \left\{ \frac{dh_2}{dt} \left(\frac{2}{b} h_2(t) - \frac{1}{b^2} h_2(t)^2\right) - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} - \dot{m}_{2u}) \right. \tag{25}$$

$$\left. - 2 \frac{r^2}{R^2} \left[\frac{2 \frac{F}{m_o (1 - a t)} (h_1(t) - h_2(t)) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} \right\} .$$

When Equations (9) and (22) are combined the result is

$$\rho \pi R^2 \left[\frac{h_1(t)^2 + h_2(t)^2}{b} - \frac{h_1(t)^3 + h_2(t)^3}{3b^2} \right] = m_{10} + m_{20} - \int_0^t (\dot{m}_{1u} + \dot{m}_{2u}) \, dt , \tag{26}$$

or

$$\frac{h_1(t)^2 + h_2(t)^2}{b} - \frac{h_1(t)^3 + h_2(t)^3}{3b^2} = \frac{1}{\rho \pi R^2} (m_{10} + m_{20}) - \frac{1}{\rho \pi R^2} \int_0^t (\dot{m}_{1u} + \dot{m}_{2u}) \, dt .$$

Differentiation of Equation (26) with respect to time gives

$$\frac{2}{b} h_{1(t)} \frac{dh_1}{dt} + \frac{2}{b} h_{2(t)} \frac{dh_2}{dt} - \frac{1}{b^2} h_{1(t)}^{\,2} \frac{dh_1}{dt} - \frac{1}{b^2} h_{2(t)}^{\,2} \frac{dh_2}{dt} = - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} + \dot{m}_{2u}) , \quad (27)$$

or with the $\frac{dh_2}{dt}$ explicit

$$\frac{dh_2}{dt} = \frac{1}{\left(\frac{2}{b} h_{2(t)} - \frac{1}{b^2} h_{2(t)}^{\,2}\right)} \left[- \frac{dh_1}{dt} \left(\frac{2}{b} h_{1(t)} - \frac{1}{b^2} h_{1(t)}^{\,2}\right) - \frac{1}{\rho \pi R^2} (\dot{m}_{1u} + \dot{m}_{2u}) \right] . \quad (28)$$

Equations (25) and (28) are the two differential equations for $h_{1(t)}$ and $h_{2(t)}$ for the lower oblate hemispheroid part of the tank. Because ordinarily the flow rates to each engine are given in the units pounds weight per second it is necessary to introduce the gravitational constant to obtain the mass flow. The same applies for converting the weight of the vehicle into mass units.

To obtain a numerical solution it is necessary to eliminate $\frac{dh_2}{dt}$ from the Equations (13), (19), and (25) for $\frac{dh_1}{dt}$. The resulting equations along with the equations for $\frac{dh_2}{dt}$ are summarized below.

<u>Upper Oblate Hemispheroid</u>

$$\frac{dh_1}{dt} = \frac{1}{\left(1 - \frac{(h_{1(t)} - K)^2}{b^2}\right)} \left\{ - \frac{\dot{\omega}_{1u}}{g_e \rho \pi R^2} - \frac{r^2}{R^2} \left[\frac{2 \frac{F \cdot g_e}{W_o(1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} \right\} , \quad (29)$$

$$\frac{dh_2}{dt} = \frac{1}{\left(1 - \frac{(h_{2(t)} - K)^2}{b^2}\right)} \left[- \frac{dh_1}{dt} \left(1 - \frac{(h_{1(t)} - K)^2}{b^2}\right) - \frac{1}{g_e \rho \pi R^2} \left(\dot{\omega}_{1u} + \dot{\omega}_{2u}\right) \right] . \tag{30}$$

Cylindrical Section

$$\frac{dh_1}{dt} = - \frac{\dot{\omega}_{1u}}{g_e \rho \pi R^2} - \frac{r^2}{R^2} \left[\frac{2 \frac{F \cdot g_e}{W_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} , \tag{31}$$

$$\frac{dh_2}{dt} = - \frac{dh_1}{dt} - \frac{1}{g_e \rho \pi R^2} \left(\dot{\omega}_{1u} + \dot{\omega}_{2u}\right) . \tag{32}$$

Lower Oblate Hemispheroid

$$\frac{dh_1}{dt} = \frac{1}{\left(\frac{2}{b} h_{1(t)} - \frac{1}{b^2} h_{1(t)}{}^2\right)} \left\{ - \frac{\dot{\omega}_{1u}}{g_e \rho \pi R^2} \right.$$

$$\left. - \frac{r^2}{R^2} \left[\frac{2 \frac{F \cdot g_e}{W_o (1 - a t)} (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} \right\} , \tag{33}$$

$$\frac{dh_2}{dt} = \frac{1}{\left(\frac{2}{b} h_{2(t)} - \frac{1}{b^2} h_{2(t)}^{\ 2}\right)} \left[- \frac{dh_1}{dt} \left(\frac{2}{b} h_{1(t)} - \frac{1}{b^2} h_{1(t)}^{\ 2}\right) - \frac{1}{g_e \rho \pi R^2} (\dot{\omega}_{1u} + \dot{\omega}_{2u}) \right]. \tag{34}$$

For the special cases when the level in tank 1 is in a different section than the level in tank 2, see Appendix B.

It is noted here that a method to keep equal fluid levels at each time is to apply a pressure difference to overcome the flow resistance in the interconnecting line. This pressure difference can be determined only by the flow rate difference between the two tanks. Using Equation (8) and setting $h_{1(t)} - h_{2(t)} = 0$ yields

$$\frac{\dot{\omega}_{2u} - \dot{\omega}_{1u}}{g_e \rho \pi R^2} = 2 \frac{r^2}{R^2} \left[\frac{2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}}, \tag{35}$$

or

$$p_1 - p_2 = \frac{(\dot{\omega}_{2u} - \dot{\omega}_{1u})^2 q}{8 \rho g_e^{\ 2} \pi^2 r^4} . \tag{36}$$

NONFLIGHT OPERATION

During nonflight operation (for example a static firing) the acceleration of the tanks is zero. Only the gravitational force is acting on the system. The equations are the same as those for the flight operation except the acceleration term under the radical in Equations 29-34 must be altered by requiring that

$$F = W_o (1 - \alpha t) .$$

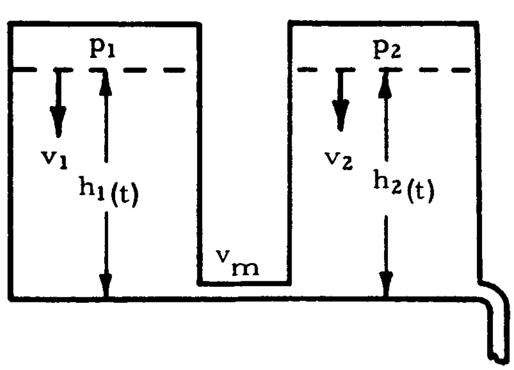
This is affected in the program by inputing

$$F = W_o = 1 \quad \text{and} \quad \dot{\omega}_p = 0 \ .$$

EMERGENCY NONFLIGHT DRAINING

During the emergency draining process only one draining tube is used to empty both tanks simultaneously. The flow rate will not be constant because the liquid heights are changing so the draining flow rate is a function of the instantaneous heights in the tanks. It has to be pointed out that a new equation is needed to determine the flow rate at each time interval. And, just as for the nonflight operation, the thrust must be set equal to $W_o (1 - \alpha t)$.

Application of the continuity equation and the Bernoulli equation gives



$$\dot{\omega}_e = v_e A_e \gamma = v_1 A_1 \gamma + v_2 A_2 \gamma$$
$$= v_m A_I \gamma + v_2 A_2 \gamma \ , \tag{37}$$

$$\frac{v_1^2}{2g} + \frac{p_1}{\gamma} + h_{1(t)} = \frac{v_m^2}{2g} + \frac{p_{I_1}}{\gamma} \ , \tag{38}$$

$$\frac{v_m^2}{2g} + \frac{p_{I_1}}{\gamma} - \frac{\Delta p_I}{\gamma} = \frac{v_m^2}{2g} + \frac{p_{I_2}}{\gamma} \ , \tag{39}$$

$$\frac{v_2^2}{2g} + \frac{p_2}{\gamma} + h_{2(t)} = \frac{v_e^2}{2g} + \frac{p_e}{\gamma} \ . \tag{40}$$

The unsteady flow term $\frac{\partial v}{\partial t}$ ds has been neglected because at relatively high tank pressure the effective influence of the decreasing liquid level height on the total head is very small and the velocity is rather constant. The velocity in the interconnecting line is

$$v_m = \left[\frac{2g\ (h_{1(t)} - h_{2(t)}) + 2\ (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} . \tag{41}$$

Applying Equations (35), (36), (37), and (38) the relation for the draining flow is, assuming the cross section areas of the tanks are equal $A_1 = A_2 = A$,

$$\frac{\dot{\omega}_e}{r} = A_I \left[\frac{2g\ (h_{1(t)} - h_{2(t)}) + 2\ (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} + A \left\{ 2g \left[\frac{\dot{\omega}_e^2}{2g A_e^2 \gamma^2} + \frac{p_e}{\gamma} - \frac{p_2}{\gamma} - h_{2(t)} \right] \right\}^{\frac{1}{2}} . \tag{42}$$

This is an implicit equation and can be easily converted into an explicit form.

$$\dot{\omega}_e = \frac{g_e \rho \pi}{\frac{R^4}{r_e^4} - 1} \left\{ R^2 \left[\frac{r^4}{r_e^4} \left[\frac{2 g_e (h_{1(t)} - h_{2(t)}) + 2(p_1 - p_2)/\rho}{q} \right] + \left(2 g_e h_{2(t)} + \frac{p_2 - p_e}{\rho} \right) \left(\frac{R^4}{r_e^4} - 1 \right) \right]^{\frac{1}{2}} \right. \tag{43}$$

$$\left. - r^2 \left[\frac{2 g_e (h_{1(t)} - h_{2(t)}) + 2 (p_1 - p_2)/\rho}{q} \right]^{\frac{1}{2}} \right\} .$$

This equation for the draining flow rate has been programmed, and has to be calculated after each time interval chosen for the main program. At the time $t = 0^+$ the draining flow rate would be a maximum of

$$\dot{\omega}_e = g_e \rho \pi R^2 \left[\frac{2 g_e h_{20} + \frac{p_2 - p_e}{\rho}}{\frac{R^4}{r_e^4} - 1} \right]^{\frac{1}{2}} . \tag{44}$$

GROUND FILLING OPERATION

For the filling operation the equations are the same as those for the inflight drain. However, the value of $\dot{\omega}_{1u}$ must have a negative sign and $\dot{\omega}_{2u}$ must be zero. In the case of filling both tanks through only one tank, $\dot{\omega}_{2u}$ should be set equal to zero. Additionally

$$F = W_o = 1 \quad \text{and} \quad \dot{\omega}_p = 0 .$$

RESULTS

Sample results are presented in this section for the SVI Stage Multi-Mission Module Lox tanks during in-flight operation, emergency draining, and filling. Graphical representation of the variables is presented for a given r and q in Figures 1, 2, and 3 for each of the above cases. These curves show the history of the liquid levels (h_1, h_2), liquid level difference ($h_1 - h_2$), weight difference (ΔW), mean velocity of the fluid in the interconnect line (v_m), and vehicle acceleration (g). In addition, Figure 4 illustrates the relationship between the weight left and r with q as a parameter. Figure 5 shows the time required for tank 2 to empty as a function of r with q as a parameter. Constants used in the calculations are given in Table 1.

Table 1

Constants used in the Calculations

$K = 90.62$ in

$R = 28.5$ in

$h_{10} = h_{20} = 94.4526$ in

$\dot{\omega}_{1u} = 27.723$ lb/sec

$\dot{\omega}_{2u} = 29.0999$ lb/sec

$F = 30,000$ lb

$\dot{\omega}_p = 68.186$ lb/sec

$p_1 = p_2 = 0$ lb/in^2

$b = 20.15$ in

$r = 3.0$ in

$\rho = 0.106106$ slugs/in^3

$W_o = 35,500$ lb

$q = 30$

Weight of initial amount of liquid = 18,333 lb

For Emergency Drain:

$p_1 = p_2 = 30$ lb/in^2

$p_e = 16.0$ lb/in^2

$r_e = 1.0$ in

$\dot{\omega}_{1u} = 0.0$ lb/sec

$F = W_o = 1$

$\alpha = 0.0$

Table 1 (CONT.)

For Filling:

$r_e = 1.0$ in

$\dot{\omega}_{1u} = -10.0$ lb/sec

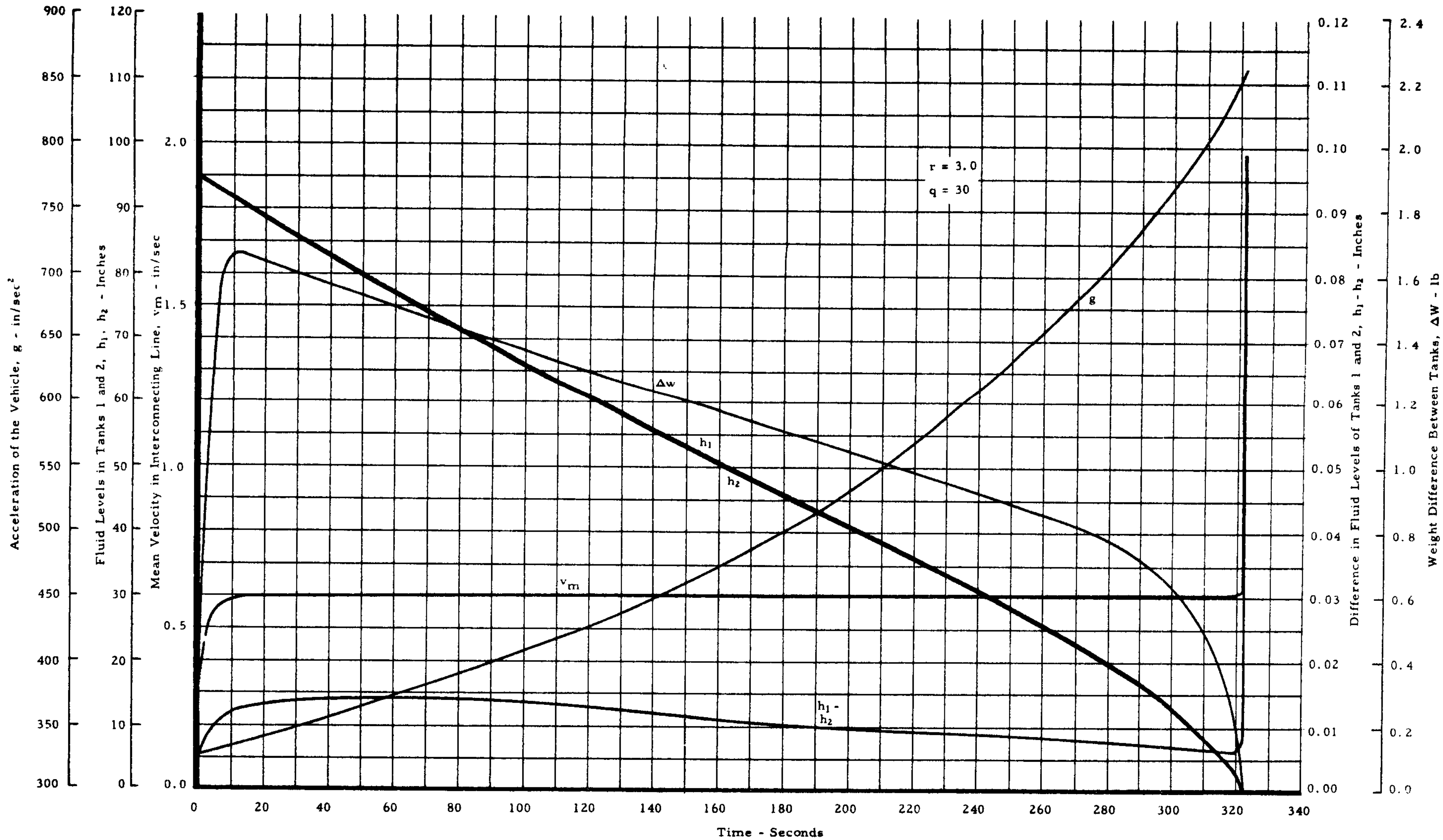


Figure 1

SVI Stage Lox Tanks - In-Flight Operation

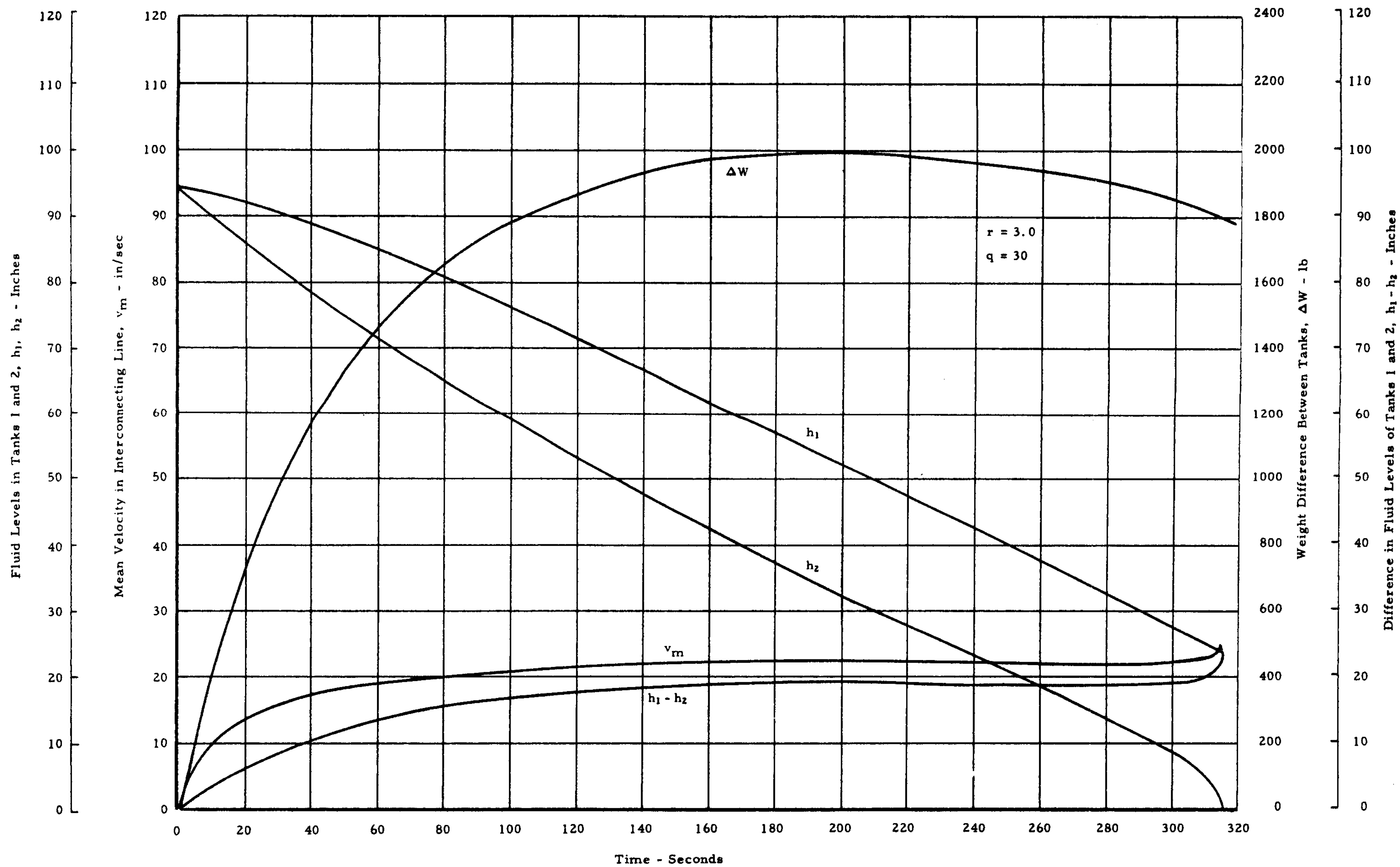


Figure 2

SVI Stage Lox Tanks - Emergency Drain

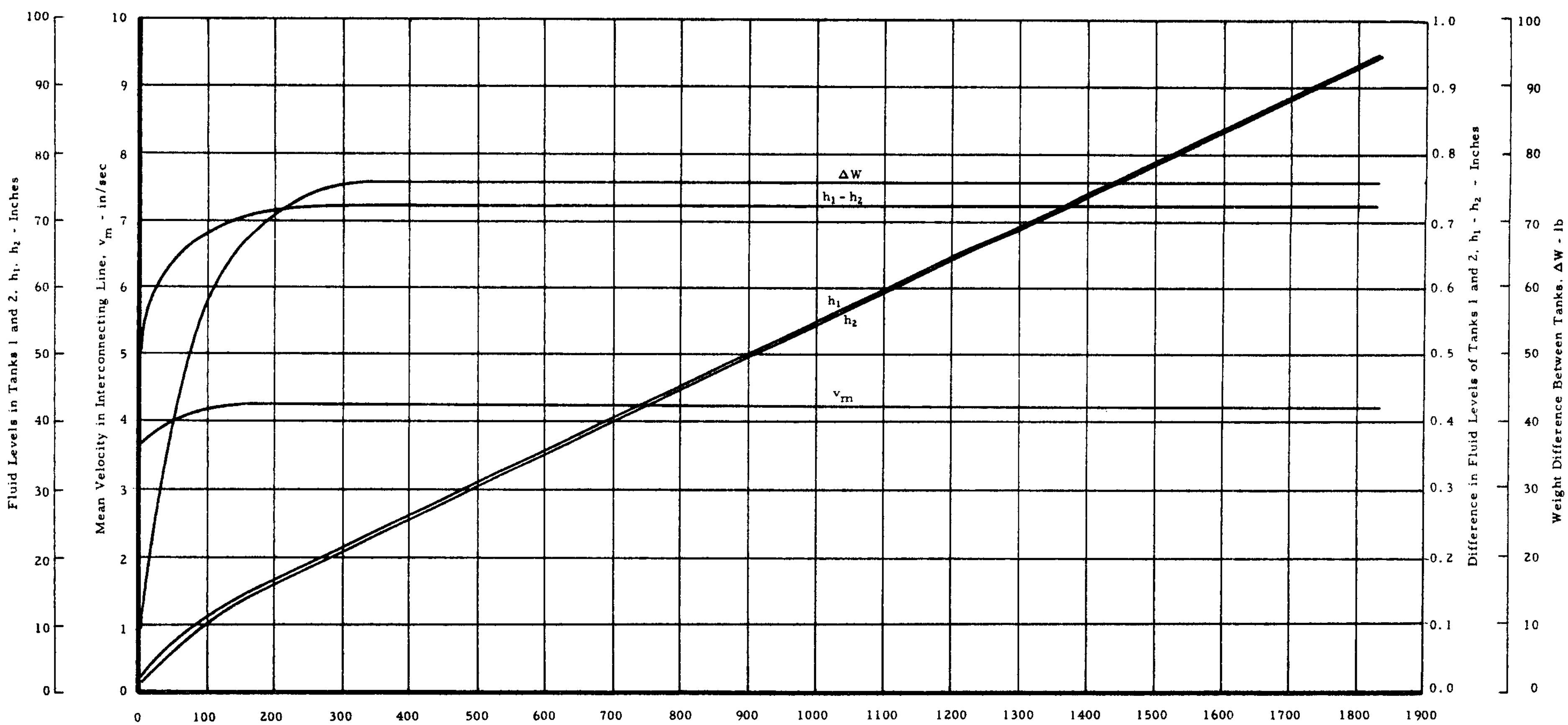


Figure 3

SVI Stage Lox Tanks - Filling Operation

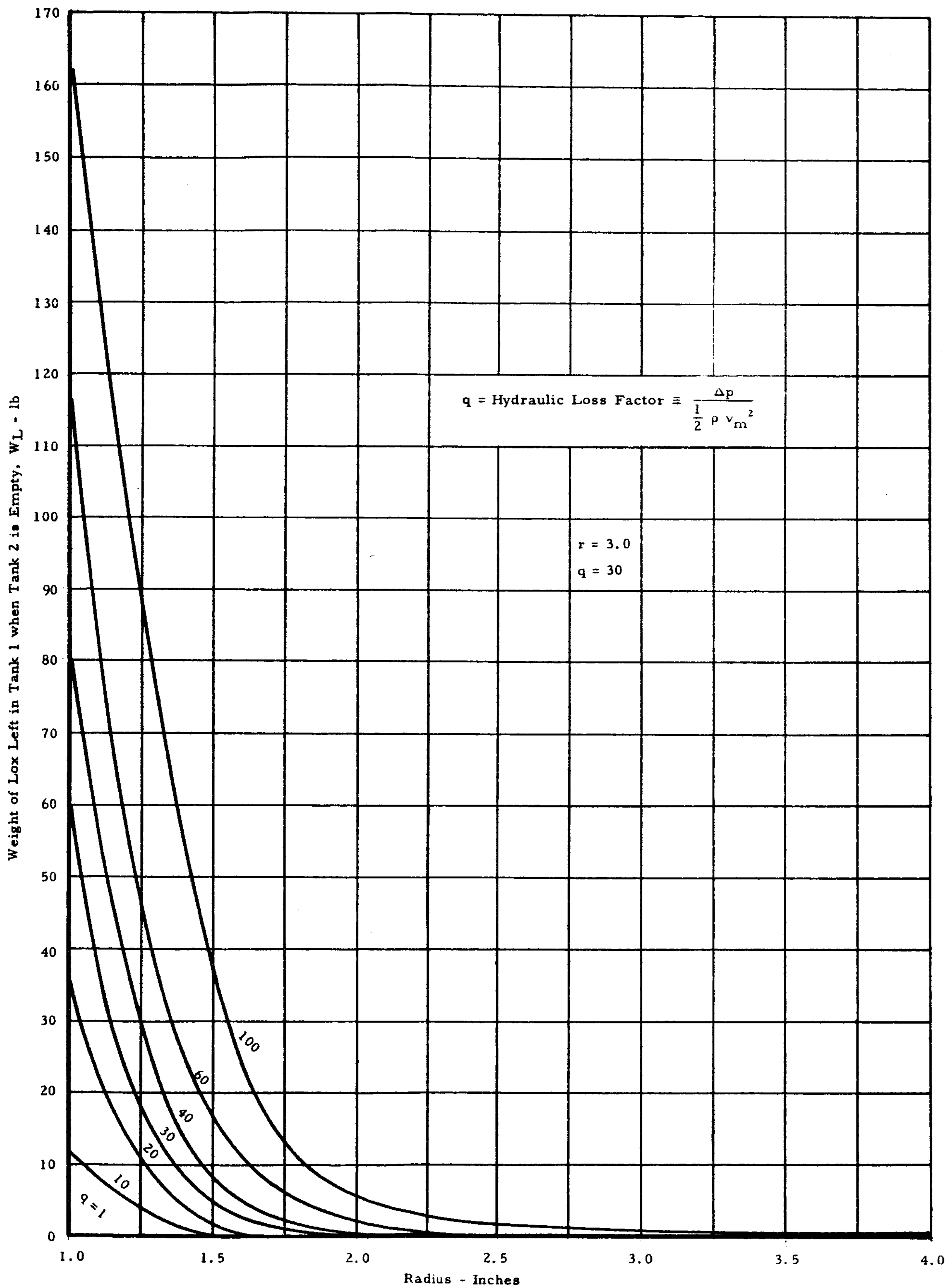


Figure 4

SVI Stage Lox Tanks - In Flight Operation
Relation Between Interconnect Line Size and Weight of Remaining Lox

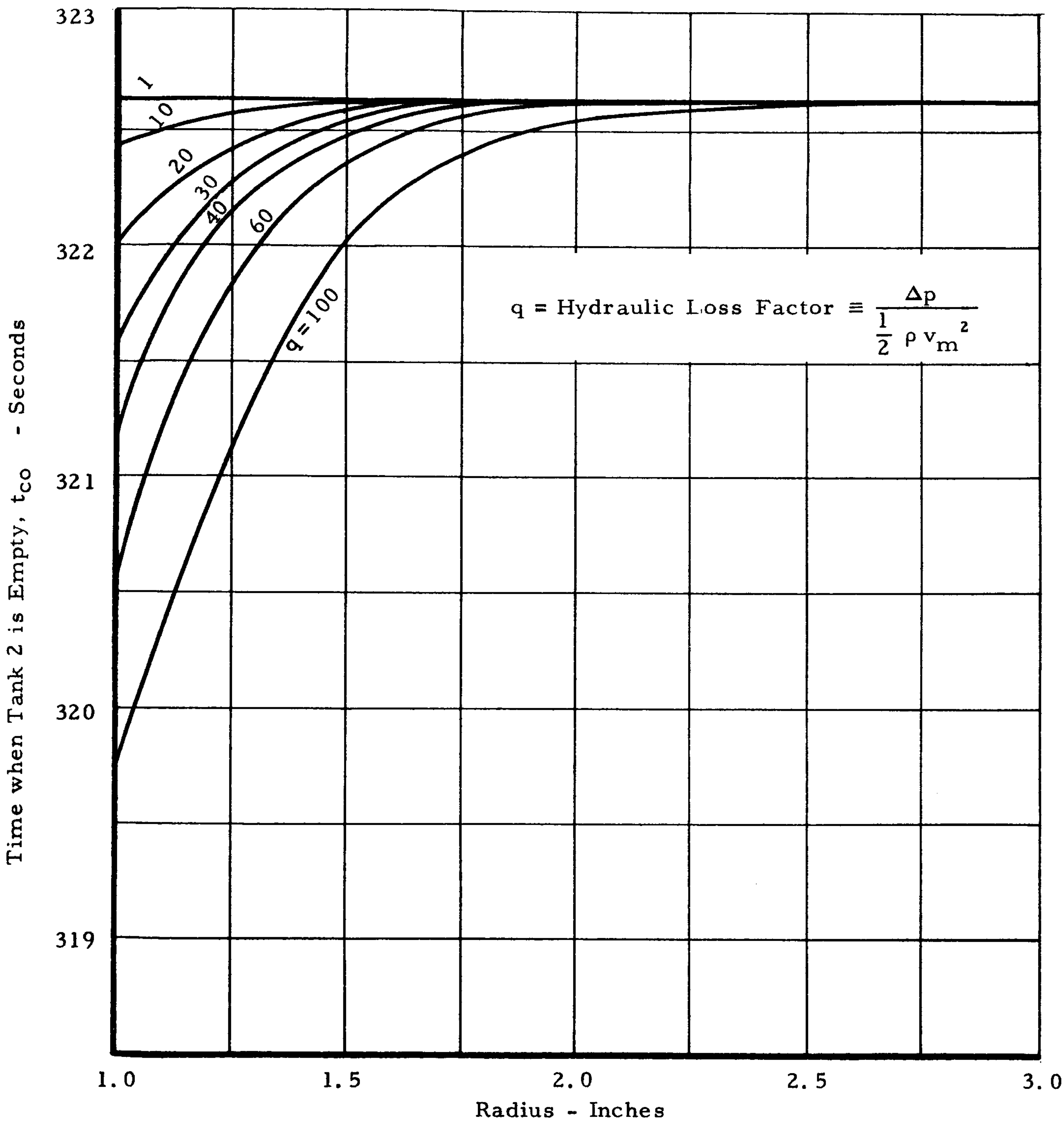


Figure 5

SVI Stage Lox Tanks - In Flight Operation
Relation Between Interconnect Line Size and Cut-Off Time.

CONCLUSIONS

The nonlinear differential equations solved by the Runge-Kutta method describe a transient incompressible fluid flow system, and the application can be enlarged to a variety of similar systems of two tanks with an interconnecting line.

REFERENCES

1. "A FORTRAN Program to Calculate the Relative Liquid Levels in Two Interconnected Tanks with Different Flow Rates", B. H. Kavanaugh, Jr., H. H. Seidel, and J. B. Cox, Brown Engineering Company Technical Note R-70, September, 1963

2. "Analysis of the Relative Liquid Levels in Two Interconnected Tanks with Different Flow Rates - Part II: Application to the Multi-Mission Module", H. H. Seidel and J. B. Cox, Brown Engineering Company Technical Note R-73, September, 1963

APPENDIX A

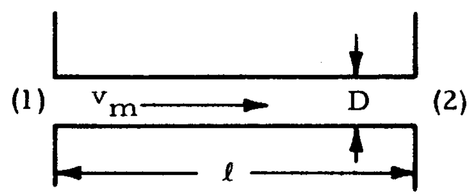
COMMENT ON THE HYDRAULIC LOSS FACTOR

The hydraulic loss factor for a tube is defined by the equation

$$q \equiv \frac{\Delta p}{\frac{1}{2} \rho v_m^2} \qquad \text{(A-1)}$$

where Δp is the pressure drop in the line. For example, in the sketch below

$$\Delta p = p_1 - p_2 \qquad \text{(A-2)}$$



There exists a simple relationship between q and the quantity h_L which is usually included in the Bernoulli equation for incompressible fluid motion in pipes to account for the head loss due to wall friction, valves, bends, etc. For the case illustrated above the head loss is given by

$$h_L = \frac{\Delta p}{\rho g} . \qquad \text{(A-3)}$$

This may also be written

$$h_L = \frac{\Delta p}{\rho g} = q \frac{v_m^2}{2g} \quad . \tag{A-4}$$

so that

$$q = \frac{2g}{v_m^2} h_L \tag{A-5}$$

The total head loss, h_L, consists of the sum of the loss due to wall friction and the minor losses. Minor losses generally include losses due to the pipe entrance and exit, bends, valves, and changes in cross section. The wall friction loss is given by

$$f \frac{\ell}{D} \frac{v_m^2}{2g}$$

where f is determined from the Moody diagram for a particular tube roughness and Reynold's number. Minor losses are accounted for by the term

$$K_L \frac{v_m^2}{2g} \quad .$$

Values for K_L must be determined experimentally. Its value depends primarily on the geometry of the loss producing element. The total head loss h_L may now be expressed as an algebraic sum of the various losses as follows

$$h_L = f \frac{\ell}{D} \frac{v_m^2}{2g} + K_{L_1} \frac{v_m^2}{2g} + K_{L_2} \frac{v_m^2}{2g} + \ldots . \tag{A-6}$$

Therefore

$$q = f\frac{\ell}{D} + K_{L_1} + K_{L_2} + \ldots \tag{A-7}$$

Each of the K_L's (K_{L_1}, K_{L_2}, ...) is for a particular valve, entrance, exit, etc.

APPENDIX B

SPECIAL CASES

When the levels in the two tanks are in different sections, the derivation of the equations for $\frac{dh_1}{dt}$ and $\frac{dh_2}{dt}$ proceeds as for the other cases. The expressions for the volume which are substituted in Equations (8) and (9) must correspond to the location of the liquid levels. The results of the derivations are presented below.

h_1 in the upper oblate hemispheroid

h_2 in the cylindrical section

$$\frac{dh_1}{dt} = \frac{1}{1 - \frac{(h_1 - K)^2}{b^2}} \left\{ - \frac{\dot{\omega}_{1u}}{g_e \rho \pi R^2} - \frac{r^2}{R^2} \left[\frac{2 \frac{F \cdot g_e}{W_o(1 - \alpha t)} (h_1 - h_2) + 2 \frac{p_1 - p_2}{\rho}}{q} \right]^{\frac{1}{2}} \right\} \tag{B-1}$$

$$\frac{dh_2}{dt} = - \left[1 - \frac{(h_1 - K)^2}{b^2} \right] \frac{dh_1}{dt} - \frac{\dot{\omega}_{1u} + \dot{\omega}_{2u}}{g_e \rho \pi R^2} \tag{B-2}$$

h_1 in the cylindrical section

h_2 in the lower oblate hemispheroid

$$\frac{dh_1}{dt} = - \frac{\dot{\omega}_{1u}}{g_e \rho \pi R^2} - \frac{r^2}{R^2} \cdot \left[\frac{2 \frac{F \cdot g_e}{W_o(1 - \alpha t)} (h_1 - h_2) + 2 \frac{p_1 - p_2}{\rho}}{q} \right]^{\frac{1}{2}} \tag{B-3}$$

$$\frac{dh_2}{dt} = \frac{1}{2 \frac{h_2}{b} - \frac{h_2^2}{b^2}} \left[- \frac{dh_1}{dt} - \frac{\dot{\omega}_{1u} + \dot{\omega}_{2u}}{g_e \rho \pi R^2} \right] \tag{B-4}$$

h_1 in the upper oblate hemispheroid

h_2 in the lower oblate hemispheroid

$$\frac{dh_1}{dt} = \frac{1}{1 - \frac{(h_1 - K)^2}{b^2}} \left\{ - \frac{\dot{\omega}_{1u}}{g_e \; \rho \; \pi R^2} - \frac{r^2}{R^2} \left[\frac{2 \frac{F \cdot g_e}{W_o (1 - at)} (h_1 - h_2) + 2 \frac{p_1 - p_2}{\rho}}{q} \right]^{\frac{1}{2}} \right\} \quad \text{(B-5)}$$

$$\frac{dh_2}{dt} = \frac{1}{2 \frac{h_2}{b} - \frac{h_2^2}{b^2}} \left\{ - \frac{dh_1}{dt} \left[1 - \frac{(h_1 - K)^2}{b^2} \right] - \frac{\dot{\omega}_{1u} + \dot{\omega}_{2u}}{g_e \rho \pi R^2} \right\} \quad \text{(B-6)}$$